

Exact microscopic wave function for a topological quantum membrane

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The higher dimensional quantum Hall liquid constructed recently supports stable topological membrane excitations. Here we introduce a microscopic interacting Hamiltonian and present its exact ground state wave function. We show that this microscopic ground state wave function describes a topological quantum membrane. We also construct variational wave functions for excited states using the non-commutative algebra on the four sphere. Our approach introduces a non-perturbative method to quantize topological membranes.

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Recently, a higher dimensional generalization of the quantum Hall effect has been constructed by Zhang and Hu (ZH)[1]. The fundamental fermionic fluid particles move on the surface of a four sphere (S^4) with radius R , and carry $SU(2)$ gauge degrees of freedom in the representation I . The instanton density of the $SU(2)$ gauge field is uniformly distributed over S^4 . ZH considered the limit where $I \rightarrow \infty$ when $R \rightarrow \infty$, such that $R^2/2I \equiv l^2$ is held constant. l defines the fundamental magnetic length in this problem. This quantum Hall liquid shares many properties of the familiar 2D quantum Hall liquid, including incompressibility, fractional charge and gapless edge states. This theory has been further developed in refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Bernevig et al[8] have recently constructed a topological field theory for the new quantum Hall liquid. The configuration space in this problem is CP_3 , which is locally the product of the orbital space S^4 and the isospin space S^2 . The Chern-Simons theory can be defined either over the configuration space CP_3 , or on the orbital space S^4 . The later can be obtained from the former through the fuzzification of the isospin sphere S^2 . This field theoretical study reveals an important class of extended topological objects, including the membrane (2-brane) and the 4-brane. The membranes wrap the isospin S^2 and have a non-trivial statistical interaction which generalize the concept of fractional statistics of Laughlin quasi-particles.

In this paper, we investigate microscopic properties of the membranes found in the study of Bernevig et al. We shall introduce a microscopic interaction Hamiltonian in the lowest-Landau-level (LLL), and find its *exact* ground state wave function. This wave function is a natural generalization of Laughlin's wave function for the 2D QHE[12, 13]. We then show that this wave function describes a collection of particles forming a membrane, wrapped around the isospin S^2 . This wave function is a $SO(5)$ singlet, therefore, the center of mass of the membrane is uniformly delocalized on S^4 . We also discuss the excitations of the membrane in terms of the non-commutative algebra on S^4 .

The importance of the exact membrane wave function

may be viewed from different perspectives. Up to this point, only microscopic information about the ZH model are based on non-interacting physics. An elementary analysis of the boundary degrees of freedom reveals an “embarrassment of riches”[1, 2]. The total configuration space at the boundary is $S^3 \times S^2$. In the non-interacting limit, isospin excitations are gapless, leading to massless states with all helicities. In addition, there is also an incoherent fermionic background. The entropy at the boundary therefore scales like R^5 , unlike the R^3 scaling one would expect from a conventional 3 + 1 dimensional field theory. A possible solution to this problem could come from an “isospin gap”, introduced by the mutual interaction among the particles. In this case, for energy scales below the “isospin gap”, the entropy would scale like R^3 . Our exact membrane solution in the presence of the interaction indeed suggests this behavior. Since our membrane wraps the isospin S^2 , its internal degrees of freedom behaves like a a 2D QH liquid with an incompressibility gap. Beyond the problem of current interest, our exact membrane solution gives a new way to quantize a membrane beyond Polyakov's path integral quantization, and could yield valuable information about the strong coupling limit of quantum membranes. Finally, two possible interpretation of the elusive “M theory” are the matrix and membrane theories[14, 15]. In our model, these two theories are intimately related. In the LLL, the fundamental particles can be described by a matrix model. Our exact wave function shows microscopically how membranes emerge from a collection of matrix particles.

Let us first recall that the single particle wave function in the LLL of the 4DQHE problem is given by

$$\sqrt{\frac{p!}{m_1!m_2!m_3!m_4!}} \Psi_1^{m_1} \Psi_2^{m_2} \Psi_3^{m_3} \Psi_4^{m_4} \quad (1)$$

with integers $m_1 + m_2 + m_3 + m_4 = p = 2I$. The four component spinors Ψ_α can be expressed directly in terms of the S^4 and the S^2 coordinates, as given in eq. (5) of ref. [1]. The degeneracy of the single particle ground

states is given by

$$D(p) = \frac{1}{6}(p+1)(p+2)(p+3) \quad (2)$$

A rather remarkable feature is that while higher LL states are $SO(5)$ symmetric, the LLL states enjoy an additional $SU(4)$ symmetry, as one can see directly from eq. (1). $SO(5)$ group is isomorphic to the $Sp(4)$ group, which differs from the $SU(4)$ group only through an additional structure associated with the charge conjugation matrix \mathcal{R} . Let Γ_a , with $a = 1, \dots, 5$ be the five Dirac Gamma matrices satisfying the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$, then $\Gamma_{ab} = -\frac{i}{2}[\Gamma_a, \Gamma_b]$ form the generators of the $SO(5)$ Lie algebra. The \mathcal{R} matrix is defined by the following properties:

$$\mathcal{R}^2 = -1, \quad \mathcal{R}^\dagger = \mathcal{R}^{-1} = {}^t\mathcal{R} = -\mathcal{R} \quad (3)$$

$$\mathcal{R}\Gamma^a\mathcal{R} = -{}^t\Gamma^a, \quad \mathcal{R}\Gamma^{ab}\mathcal{R} = {}^t\Gamma^{ab} \quad (4)$$

The relation $\mathcal{R}\Gamma^{ab}\mathcal{R}^{-1} = -(\Gamma^{ab})^*$ indicates that the spinor representation of $SO(5)$ is pseudo-real. The \mathcal{R} matrix plays a role similar to that of $\epsilon_{\alpha\beta}$ in $SO(3)$. In the explicit representation given by eq. (4) of ref. [1], the \mathcal{R} matrix takes the form

$$\mathcal{R} = -i \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix} \quad (5)$$

where σ_y is a Pauli matrix. The presence of the \mathcal{R} matrix breaks the $SU(4)$ symmetry down to the $SO(5) = Sp(4)$ symmetry. Since the LLL wave functions do not involve the \mathcal{R} matrix, they are $SU(4)$ invariant. However, the \mathcal{R} matrix is needed to construct wave functions in higher LLs[16], which are only $SO(5)$ invariant.

Having reviewed the wave functions in the LLL and introduced the concept of the \mathcal{R} matrix, we now present our microscopic wave function. In the 2DQHE, the $\nu = 1$ wave function can be expressed either as a Slater determinant or as a Jastrow-Laughlin type product wave function. The van der Monde identity relates them exactly. However, this identity does not hold in higher dimension. We shall see the profound physical implications introduced by this inequivalence. In ref. [1], ZH constructed many-body wave functions by using the Slater determinant, and following Laughlin, by taking odd powers of these Slater determinant. They showed that these wave functions describe incompressible quantum liquids. However, one could proceed in a different way here, by constructing wave functions using the inequivalent Jastrow-Laughlin product form. Such a wave function takes the form

$$\Phi_0 = \prod_{i < j} (\Psi_\alpha(i)\mathcal{R}^{\alpha\beta}\Psi_\beta(j))^m \quad (6)$$

where m is an odd integer and i, j refers to i th and j th particles in the system. If we replace $\mathcal{R}^{\alpha\beta}$ by $\epsilon^{\alpha\beta}$,

and let α, β to take values of 1, 2, this would transform exactly into Laughlin's wave function expressed in Hal-dane's spherical geometry[13]. Our wave function has the following properties:

1) When m is an odd integer, this wave function is antisymmetric when particle coordinates are exchanged. Therefore, this wave function describes a fermionic system.

2) The wave function is a $SO(5)$ singlet. This is because every term in the product, $\Psi_\alpha(i)\mathcal{R}^{\alpha\beta}\Psi_\beta(j)$ is a $SO(5)$ singlet, by virtue of eq. (4).

Since the wave function involves the \mathcal{R} matrix explicitly, the symmetry in the LLL is broken from $SU(4)$ down to $SO(5)$.

3) When the product is expanded, the spinor coordinate $\Psi(i)$ occurs $m(N-1)$ times. Therefore, the wave function for the i th particle takes the form of (1), with $p = m(N-1)$. ZH showed that single particle level spacing becomes finite in the limit when $p/R^2 = 1/l^2$ is held constant. Therefore, the number of particles in the wave function (6) scales like $N \sim p \sim R^2$. In other words, *the wave function (6) describes a two dimensional object*.

This wave function can be represented graphically. We associate each particle with p dots, representing a symmetric spinor state of the form of (1). We draw a solid line representing a *contraction* between the i th and the j th particle through the \mathcal{R} matrix. The resulting graphical representation for $N = 4, m = 1, 3$ is depicted in Fig. 1a and Fig. 1b.

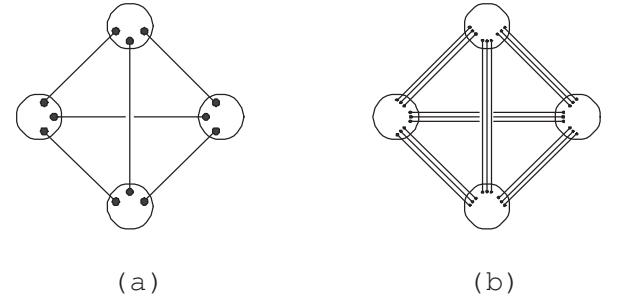


FIG. 1: A graphical representation of the wave function (6), when the product is expanded. Each particle is denoted by a circle with p dots, representing a LLL state $(p, 0)$. Each solid line denotes a $SO(5)$ singlet bond formed between the spinor indices of particles i and j .

4) In order to see what kind of two dimensional object is described by the correlated wave function (6), we borrow from Laughlin's plasma analogy. $\int \prod_i dX_i dn_i |\Phi_0|^2$ can be interpreted as the partition function of a classical gas, living in the six dimensional CP_3 configuration space. The Boltzmann weight for this classical gas is

$$|\Phi_0|^2 = \prod_{i < j} |\Phi(ij)|^{2m} = e^m \sum_{i < j} \log |\Phi(ij)|^2 \quad (7)$$

where $\Phi(ij) = \Psi_\alpha(i)\mathcal{R}^{\alpha\beta}\Psi_\beta(j)$. Since the wave function involves only pair-wise correlations, we see that the classical gas interacts via a two-body potential only. m can be interpreted as the effective inverse temperature, $\beta = 1/kT$. Using the tensor identity

$$\mathcal{R}_{\alpha\beta}\mathcal{R}_{\gamma\delta} = \frac{1}{4}\delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{1}{4}\Gamma_{\alpha\gamma}^a\Gamma_{\beta\delta}^a - \frac{1}{4}\Gamma_{\alpha\gamma}^{ab}\Gamma_{\beta\delta}^{ab} \quad (8)$$

we obtain the explicit form of the pair-wise potential

$$|\Phi(ij)|^2 = \frac{1}{4} + \frac{1}{4}X_a(i)X_a(j) - \frac{1}{4}F_{ab}^\alpha(i)n_\alpha(i)F_{ab}^\beta(j)n_\beta(j) \quad (9)$$

here $X_a(i)$, with $X_a^2 = 1$, describes the coordinate of the i th particle on the orbital space S^4 , and $n_\alpha(i)$, with $n_\alpha^2 = 1$, describes its coordinate on the isospin space S^2 . F_{ab}^α is the Yang-Mills field strength of the $SU(2)$ instanton over S^4 , explicitly given by eq. (19) of ref. [8].

From the sign of the second terms in eq. (9), we see that the gas particles interact via an *attractive* interaction on S^4 . This is in direct contrast to Laughlin's plasma, where the gas particles interact via a *repulsive* interaction on S^2 . Therefore, at low temperature, our gas particles have a natural tendency to cluster to the same point on S^4 . However, over every point on S^4 , there is also a large, internal isospin S^2 for the gas particles to "live". From the sign of the third term in eq. (9), we see that the gas particles *repel* each other on the isospin S^2 , just like the case of the Laughlin plasma. From these observations, we see that our gas particles cluster to the same point on S^4 , but *uniformly* fill the isospin S^2 like a 2D QH liquid. Viewed from the whole CP_3 point of view, the gas particles form a 2D *membrane*, wrapped around the isospin sphere S^2 . The center-of-mass of the membrane is a point on S^4 , our wave function consists of an equal weight linear superposition of the center of mass position over S^4 , therefore, the ground state is a $SO(5)$ singlet.

Having exhibited the key properties of our wave function and its classical plasma analog, we now present a microscopic quantum interaction Hamiltonian for which it is an exact ground state. Our construction follows closely Haldane's pseudo-potential formalism[13]. Since every particle is in the $(p, 0)$ irreducible representation (irreps) of $SO(5)$, the total $SO(5)$ quantum number of a pair of particles i and j is *generically* given by

$$(p, 0) \bigotimes (p, 0) = \sum_{k=0}^p \sum_{l=0}^k (k+l, k-l) \quad (10)$$

On the other hand, in our wave function (6), there are already m $SO(5)$ singlet contractions between the particle i and j . Therefore, the total $SO(5)$ quantum number between these two particles *in our wave function* can only be contained in $(p-m, 0) \bigotimes (p-m, 0)$. Based on this observation, we introduce the projector Hamiltonian in

the LLL as $H = \sum_{i < j} H_{ij}$, where

$$H_{ij} = \sum_{k=p-m+1}^p \sum_{l=0}^k \lambda(k+l, k-l) P_{ij}(k+l, k-l) \quad (11)$$

here $P_{ij}(k+l, k-l)$ is a projection operator onto the $SO(5)$ irreps $(k+l, k-l)$ between a pair of particles i and j and $\lambda(k+l, k-l) > 0$ is the interaction parameter in a given channel. This Hamiltonian operates fully within the LLL and is positive definite. Since a pair of particles in our wave function can not have any of the $SO(5)$ irreps specified in (11), it is annihilated by all projectors. Therefore, we have proven that our wave function (6) is an eigenstate of the Hamiltonian (11) with zero eigenvalue. Since the Hamiltonian is also positive definite, our wave function must therefore be a ground state. Our experience leads us to conjecture that this is also an unique ground state in the spherical geometry.

Having shown the exact ground state wave function (6) of the interacting Hamiltonian (11), we now proceed to discuss the excited states. Following Feynman's construction of elementary excitations with liquid helium[17], we try to construct variational states of the form:

$$\Phi = \prod_i F(X_i, n_i) \Phi_0 \quad (12)$$

where $F(X)$ is a single particle wave function over S^4 . In Feynman's case, he simply took $F(x)$ to be plane waves, and his wave function describes the center-of-mass motion of a correlated quantum liquid with finite momentum. In our case, one could use the spherical harmonics over S^4 for $F(X)$:

$$F(X) = \sum_{L=l_1, \dots, l_5} f_L X^L, \quad X^L \equiv X_1^{l_1} X_2^{l_2} X_3^{l_3} X_4^{l_4} X_5^{l_5} \quad (13)$$

Here $l_1 + \dots + l_5 = l$, and f_L is chosen such that F belongs to the fully symmetric traceless tensor representation (l, l) of the $SO(5)$ group. We argued earlier that the membrane wave function Φ_0 is a $SO(5)$ singlet, which means that the center-of-mass of the membrane has the lowest $SO(5)$ angular momentum on S^4 . The more general wave function Φ given in (12) describes higher angular momentum of the center-of-mass on S^4 .

However, there is a serious problem with the function $F(X)$. Since $X_a = \bar{\Psi}\Gamma_a\Psi$, $F(X)$ depends both on $\bar{\Psi}$ and Ψ . But eq. (1) shows that the single particle wave functions in the LLL can only involve Ψ but not $\bar{\Psi}$. The solution of this problem is provided by Girvin, MacDonald and Platzman[18]. One simply needs to use the projection of X_a in the LLL, which is

$$X_a = \frac{1}{p}\Psi\Gamma_a\frac{\partial}{\partial\Psi} \quad (14)$$

The effect of X_a operating on a $SO(5)$ singlet bond formed by \mathcal{R} is to turn it into a vector bond $\mathcal{R}\Gamma_a$. After the projection, X_a 's become operators and no longer commute with each other, in fact, they satisfy the non-commutative algebra outlined in [1]. As a consequence of the non-commutativity, $F(X)$ does not only give the fully symmetric traceless tensor representation (l, l) of the $SO(5)$ group, but includes all $SO(5)$ irreps in the series $\underline{\mathbf{5}} \otimes \underline{\mathbf{5}} \otimes \underline{\mathbf{5}} \dots$ A physical interpretation of this result is that because of the non-commutative geometry in the LLL, the center-of-mass degrees of freedom are coupled to the internal membrane degrees of freedom. A full calculation of the variational energies for the wave function (12), with F given by (13) and (14) will be carried out in the future, possibly with the assistance of numerical calculations.

However, even without explicit calculations, we can anticipate the result based on our experience with the 2D QHE. We argued before that the our membrane wrapping the isospin S^2 is made out of a 2D QH liquid, which has an incompressibility gap. Therefore, it appears likely that our quantum membrane does not have the “spike instability” of a classical membrane[15], where arbitrarily long spikes can be created at low energies. This picture has important implications on the relativistic edge dynamics of the 4DQH liquid. In the non-interacting problem, the entropy at the edge scale like $R^3 \times R^2 = R^5$, since the internal isospin excitations are gapless. In this work, we have seen that interaction can introduce a high degree of correlation. In the strong coupling limit, there are no free particle excitations, only correlated membrane excitations. Furthermore, the membranes wrap the isospin S^2 by forming a 2D QH liquid, which has an incompressibility gap. In this case, for energies below the incompressibility gap, the effective entropy at the edge would scale like R^3 . *This effect gives a mechanism of “dynamical dimensional reduction”*. Different internal membrane excitations appear as different helicity states in the $3 + 1$

dimensional world view. Therefore, higher helicity states would naturally acquire an energy gap, as a result of the interaction and quantum correlations built into the membrane wave function. However, we do not yet know a natural mechanism within our framework to gap only states with helicities greater than three. Nonetheless, we believe that the exact membrane wave function represented in this paper provides a key step towards understanding the strong correlation effects in the 4D QHE model.

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